The Fundamental Theorem II

We give an extension of the Erdős-Stone often called the Erdős-Stone-Simonovits theorem or the fundamental theorem of extremal graphs.

Theorem 1 (Erdős-Stone 1946, Erdős-Simonovits, 1966)**.** *If F is a graph with chromatic number $\chi(F)$, then*

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1}\right)\frac{n^2}{2} + o(n^2).$$

Proof. Let F be a graph with $\chi(F) = k + 1$. It is enough to prove that

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n, F)}{n^2} = \frac{1}{2}\left(1 - \frac{1}{k}\right).$$

1: Find an F-free graph that matches the bound. On the other hand, also show that if the limit is bigger by ϵ, then one can find something, where F is a subgraph.

Solution: Observe that the Turán graph $T_k(n)$ is k-chromatic and thus does not contain F and furthermore

$$e(T_k(n)) \sim \left(1 - \frac{1}{k}\right)\frac{n^2}{2}$$

which gives the lower bound.

For the upper bound let us assume (for the sake of contradiction) that there is an $\epsilon > 0$ and arbitrarily large graphs G that are F-free and have

$$\frac{e(G)}{n^2} > \frac{1}{2}\left(1 - \frac{1}{k}\right) + \epsilon.$$

Then by the Erdős-Stone theorem there is a large enough G that contains a complete multipartite graph with $k + 1$ classes each of size $|V(F)|$. Clearly F is a subgraph of such a complete multipartite graph which is a contradiction.

□

Theorem 2. *The unique extremal graph for the dodecahedron is a K_5 connected to every vertex of a $T_2(n-5)$.*

Theorem 3. *For n large enough, the extremal graph for the octahedron is a $T_2(n)$ with one class containing a matching and the other class is C_4-free.*

The structure of extremal graphs

The next theorem describes exactly what graphs have $T_k(n)$ as their extremal graph. We need the following definition: an edge e in a graph F is called **color critical** if $\chi(F - e) < \chi(F)$, i.e. if the removal of e reduces the chromatic number of the graph.

Theorem 4 (Critical edge theorem, Simonovits)**.** *Let F be a $(k+1)$-chromatic graph. For n large enough, the (unique) extremal graph for F is $T_k(n)$ if and only if F has a color critical edge.*

Obviously K_{k+1} has a color critical edge (all edges are color critical in a complete graph), so the critical edge theorem implies Turán's theorem (for large n).

Proof. **2:** First let us begin with a graph F with no color critical edge. Suppose for contradiction that $T_k(n)$ is an extremal graph and find the contradiction.

Solution: If $T_k(n)$ is an extremal graph for F, then $T_k(n)$ plus any edge contains F. Thus, F minus that edge is a subgraph of $T_k(n)$ and thus is k-chromatic, i.e., F has a color critical edge; a contradiction.

The following claim (left as an exercise) will be helpful.

Claim 5. *Let $t = 2|V(F)|$. And consider the F-free graph formed by $K_k[t]$ and an extra vertex x. The degree of x is at most $(k-1)t$ with equality in the case when x is connected to all vertices of $K_k[t]$ except for those of one class.*

Now let us suppose that F has a color critical edge and let G_n be a sequence of extremal graphs for F (there may be more than one candidate for individual values of n; pick any one).

Idea: Show, that G_n is k-colorable (for large n). And do it by finding $K_k[t]$, show that every other vertex satisfies Claim 5 and there will come the coloring. We will do it by counting a tricky difference of $D(n)$.

Define
$$D(n) = e(G_n) - e(T_k(n)).$$

The graph F is $(k+1)$-chromatic, so it is not contained by $T_k(n)$, so $e(G_n) \geq e(T_k(n))$. Thus, by the Erdős-Stone theorem we have that for n large enough G_n contains $K_k[t]$.

3: Show that the $K_k[t]$ is actually induced, not just a subrgaph.

Solution: Use the critical edge

Put $S = G_n - K_k[t]$ and denote the number of edges between S and $K_k[t]$ by $e(S, K_k[t])$. By Claim 5 we have $e(S, K_k[t]) \leq (n - kt)(k-1)t$. Now
$$e(G_n) = e(K_k[t]) + e(S, K_k[t]) + e(S).$$

4: Clearly $T_k(n)$ contains a $K_k[t]$ so write $T_k(n)$ in a similar fashion as $e(G_n)$.

Solution: Clearly $T_k(n)$ contains a $K_k[t]$ and $T_k(n) - K_k[t]$ is simply $T_k(n - kt)$. Thus
$$e(T_k(n)) = e(K_k[t]) + (n - kt)(k-1)t + e(T_k(n - kt)).$$

5: Now using the above two equations for $e(G_n)$ and $e(T_k(n))$ and the fact that $e(S) \leq e(G_{n-kt})$ (why?) show that $D(n-kt) - D(n) \geq 0$.

Solution: we get that

$$D(n-kt) - D(n) \geq (n-kt)(k-1)t - e(S, K_k[t]) \geq 0.$$

6: Therefore, for all n large enough we have $D(n-kt) = D(n)$. Why?

Solution: Looks like $D(n)$ has an upper bound and it is not increasing.

This implies that $(n-kt)(k-1)t = e(S, K_k[t])$ and thus by Claim 5 each vertex in S is adjacent all vertices of $K_k[t]$ except those in one class.

7: Finish the proof by coloring G_n and make some argument to show $G_n = T_k(n)$.

Solution: Now color the vertices of G_n according to which class of $K_k[t]$ they are not adjacent. Observe that this coloring is proper as otherwise G_n contains F. Thus, we have that G_n is k-colorable and by maximality we have $G_n = T_k(n)$.

□

Every edge in a clique is color-critical, so the Critical edge theorem gives Turán's theorem for large n. Furthermore, every edge in an odd cycle is color-critical, so we get the following corollary.

Corollary 6. *For n large enough, the unique extremal graph for the odd cycle C_{2k+1} is $T_2(n)$. In particular,*

$$\mathrm{ex}(n, C_{2k+1}) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$